Lecture notes for Abstract Algebra: Lecture 20

1 Maximal ideals and prime ideals

We deal in this section with two very important types of ideals: **the prime ideals** and **the maximal ideals**.

Definition 1. Let R be a ring:

(1) Maximal ideal: A **proper ideal** $I \subset R$ is called a **maximal ideal** if there exists no **other proper ideal** J with $I \subset J$.

$$I \subset J, \quad J \subset R \quad \text{ideal} \quad \Rightarrow \quad J = R.$$

(2) Prime ideal: A **proper ideal** $I \subset R$ is called a **prime ideal** if for any a and b in R, if $a \cdot b$ is in I, then at least one of a and b is in I.

$$\forall a, b \in R, \quad a \cdot b \in I \quad \Rightarrow \quad a \in I \quad \text{or} \quad b \in I.$$

Theorem 2. *Let R be a commutative ring with identity and I an ideal in R. Then I is a maximal ideal of R if and only if R/I is a field.*

Proof. Let M be a maximal ideal in R. If R is a commutative ring, then R/M must be a commutative ring. Clearly, $1 + M$ acts as an identity for R/M . We must show that every nonzero element in the quotient R/M has an inverse. If $a + M$ is a nonzero element in R/M , then $a \notin M$. DefineI I to be the set $\{ra + x \,|\, r \in R \text{ , } x \in M\}$. We will show that I is an ideal in R. The set I is non empty since $0a + 0 = 0$ is in I. If we have two elements $r_1 a + x_1$ and $r_2 a + x_2$ are two elements in I, then

$$(r_1 a + x_1) - (r_2 a + x_2) = (r_1 - r_2)a + (x_1 - x_2)$$

is in I. Also, for any $r \in R$ it is true that $rI \subset I$ hence, I is closed under multiplication and satisfies the necessary conditions to be an ideal. Therefore, I is an ideal properly containing M. Since M is a maximal ideal, $I = R$; consequently, by the definition of I there must be an $x \in M$ and an element $b \in R$ such that $ab + x = 1$ and

$$1 + M = ab + M = (a + M)(b + M) = ba + M.$$

Conversely, suppose that M is an ideal and R/M is a field. Since R/M is a field, it must contain at least two elements: $0 + M = M$ and $1 + M$. Hence, M is a proper ideal of R. Let I be any ideal properly containing M. We need to show that $I = R$. Choose a in I but not in M . Since $a + M$ is a nonzero element in a field, there exists an element $b + M \in R/M$ such that $(a+M)(b+M) = ab+M = 1+M$. Consequently, there exists an element $x \in M$ such that $ab+x = 1$ and $1 \in I$. Therefore, $r \cdot 1 = r \in I$ for all $r \in R$. Consequently, $I = R$. $\square$

Proposition 3. *Let R be a commutative ring with identity 1, where $1 \neq 0$. Then P is a prime ideal in R if and only if R/P is an integral domain.*

Proof. First let us assume that P is an ideal in R and R/P is an integral domain. Suppose that $a \cdot b \in P$. If $a + P$ and $b + P$ are two elements of R/P such that $(a + P)(b + P) = 0 + P = P$, then either $a + P = P$ or $b + P = P$. This means that either $a \in P$ or $b \in P$, which shows that P must be prime.
Conversely, suppose that P is prime and $(a+P)(b+P) = ab+P = 0+P = P$. Then $a \cdot b \in P$. If $a \notin P$, then b must be in P by the definition of a prime ideal; hence, $b + P = 0 + P$ and R/P is an integral domain. □

Corollary 4. *Every maximal ideal in a commutative ring with identity is also a prime ideal.*

Example 5. Every ideal in $\mathbb{Z}$ is of the form $n\mathbb{Z}$. The factor ring $\mathbb{Z}/n\mathbb{Z}$ is an integral domain only when n is prime. It is actually a field. Hence, the nonzero prime ideals in $\mathbb{Z}$ are the ideals $p\mathbb{Z}$, where p is prime. This example really justifies the use of the word "prime" in our definition of prime ideals.

Example 6. (Ring of polynomials in two variables) Let K be a field and $K[x, y]$ the ring of polynomials in two variables x and y. That is:

$$K[x, y] = \{\sum_{n=0}^{k} \sum_{m=0}^{l} a_{n,m} x^n y^m \mid a_{n,m} \in K\} = K[x] + yK[x, y].$$

Consider the ideal $I = \langle y \rangle$ generated by the polynomial y. The quotient ring

$$K[x, y]/I = K[x, y]/\langle y \rangle = (K[x] + yK[x, y])/y = K[x]$$

is an integral domain (since K is a field) but not a field (since most polynomials do not have inverse in $K[x]$. As a consequence, the ideal $I = \langle y \rangle$ of $K[x, y]$ **is prime but not maximal**. On the other hand, in the ring of polynomials $K[x]$ in one variable (with a field K), the maximal and prime ideals coincide!